Section 5.6: Area between two curves

Math 1552 lecture slides adapted from the course materials
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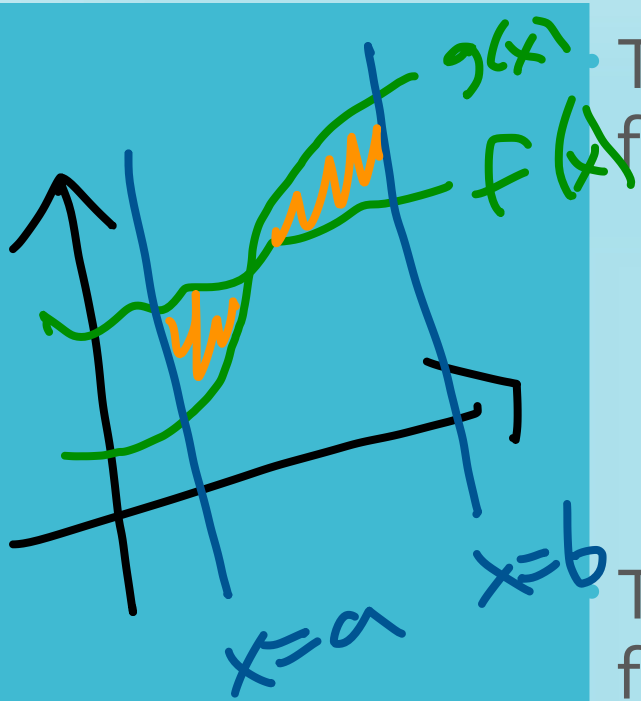
Today's Learning Goals

- Understand what is meant graphically by integrating the difference between two functions (*solve for intersection points between the two curves on the interval*)
- Set up an integral to find the total area bounded between two curves
- Evaluate numerically the area bounded between two curves
- Be able to express the integration in terms of either x or y, depending on the function(s)

Area Between Two Curves

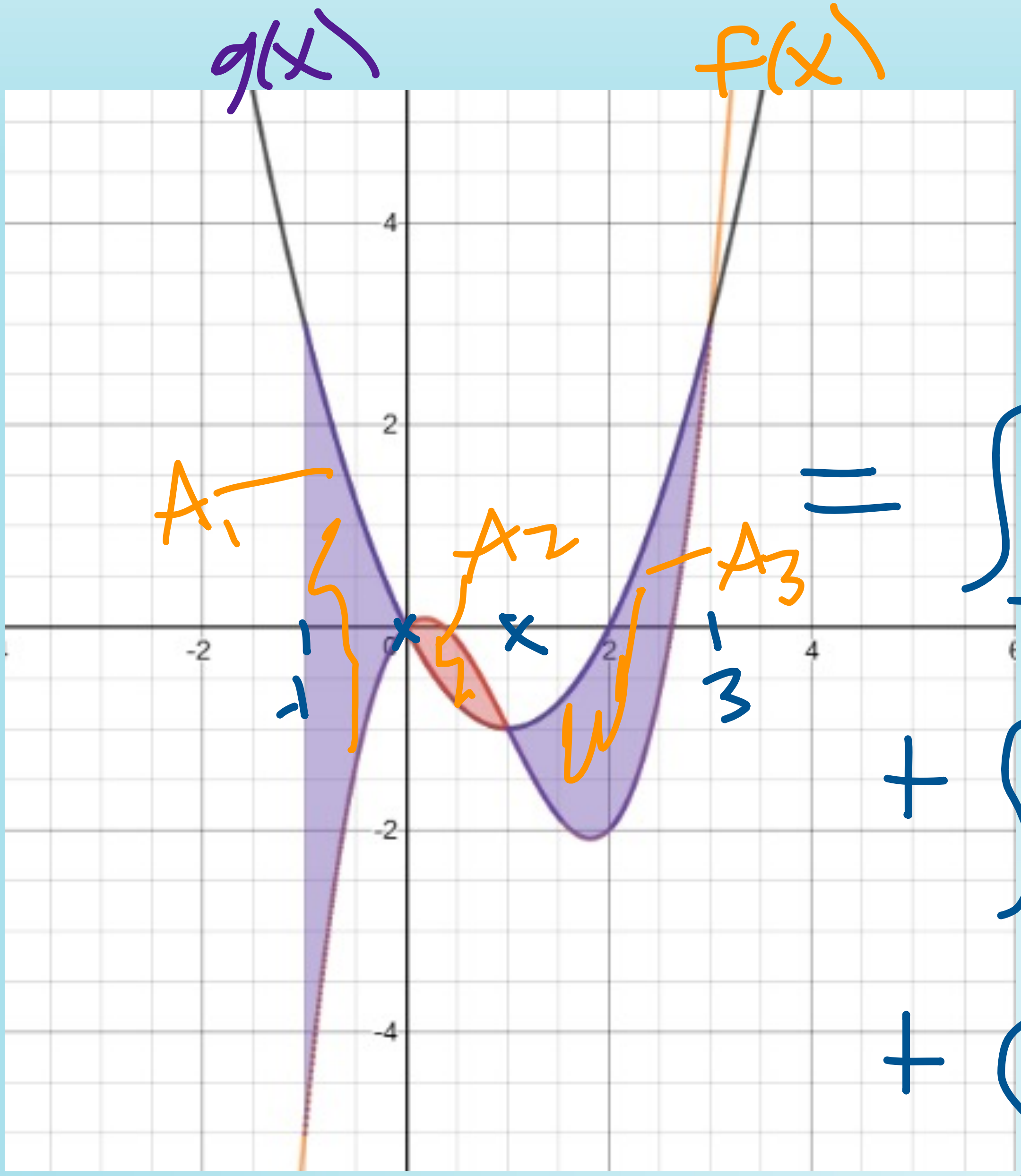


- To find the area between two curves, written as functions of *x*:

$$A=\int_a^b |\,f(x)-g(x)\,|\,dx=\int_a^b (top-bottom)dx$$

- To find the area between two curves, written as functions of *y*:

$$A=\int_{\cancel{a}\;c}^{\cancel{b}\;d} |\,f(y)-g(y)\,|\,dy=\int_{\cancel{a}\;c}^{\cancel{b}\;d} (right-left)dy$$



$$\int_{-1}^{3} |f(x)-g(x)|\,dx = \underbrace{\int_{-1}^{0}(g(x)-f(x))\,dx}_{A_1} + \underbrace{\int_{0}^{1}(f(x)-g(x))\,dx}_{A_2} + \underbrace{\int_{1}^{3}(g(x)-f(x))\,dx}_{A_3}$$

Steps to Evaluating Area

1. Where do the curves intersect? Break up the interval $[a,b]$ into sub-intervals based on points of intersection.
2. For each subinterval, which function is bigger?
3. Integrate *top-bottom* or *right-left* on each subinterval.

Example 1:

Find the area bounded by

the curves $y_1 = -x^2 + 2x - 3$ and $y_2 = x^2 - 4x + 1$

and the lines $x = 1$ and $x = 3$.

① find the points of intersection

(set $y_1 = y_2$)

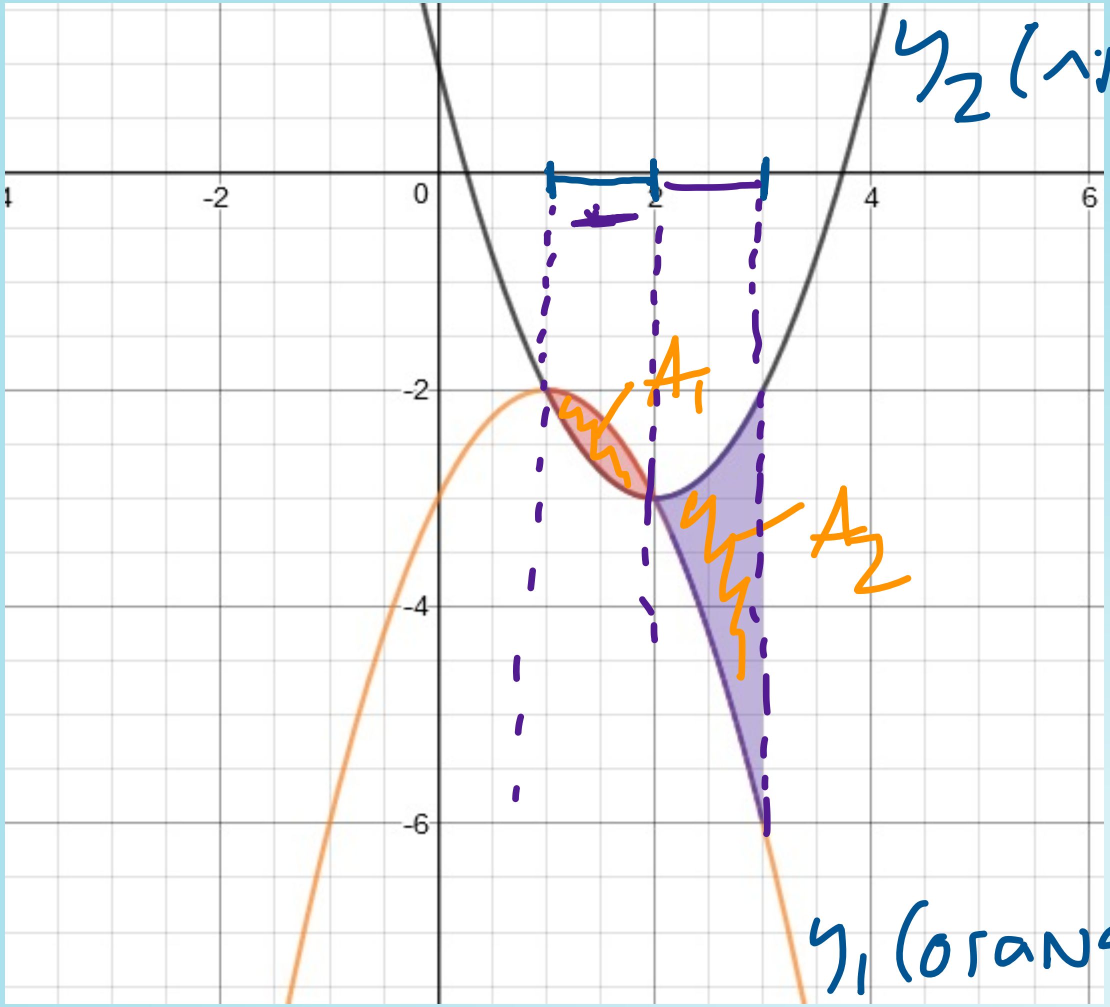
$-x^2 + 2x - 3 = x^2 - 4x + 1$

$\Leftrightarrow 2x^2 - 6x + 4 = 0$

$\Leftrightarrow x^2 - 3x + 2 = 0$

$(x-1)(x-2) = 0 \longrightarrow$ int pts: $x = +1, +2$

$y_1 = -x^2 + 2x - 3$ and $y_2 = x^2 - 4x + 1$ ON $[1, 3]$



y_2 (in gray)

y_1 (orange)

Step 2:

- ON $[1,2]$, $y_1 \geq y_2$
- ON $[2,3]$, $y_2 \geq y_1$

want to find $A = A_1 + A_2 = \boxed{2}$

Step 3: $A = \int_1^2 (y_1(x) - y_2(x))\,dx$

$+ \int_2^3 (y_2(x) - y_1(x))\,dx$

$y_1 = -x^2 + 2x - 3$ and $y_2 = x^2 - 4x + 1$ ON $[1, 3]$

$\Rightarrow A = \int_1^2 (-2x^2 + 6x - 4)\,dx$

$+ \int_2^3 (2x^2 - 6x + 4)\,dx$

$$\left.\begin{aligned} &= \left(-\frac{2}{3}x^3 + 3x^2 - 4x\right)\Big|_1^2 \\ &\quad - \left(-\frac{2}{3}x^3 + 3x^2 - 4x\right)\Big|_2^3 \end{aligned}\right] \text{apply FTC from Friday}$$

$$= \left[\left(-\frac{16}{3} + 12 - 8\right) - \left(-\frac{2}{3} + 3 - 4\right)\right]$$

$$- \left[\left(-18 + 27 - 12\right) - \left(-\frac{16}{3} + 12 - 8\right)\right]$$

$\longrightarrow$ simplify

$$= \left[-\frac{14}{3} + 4 + 1\right] - \left[-3 + \frac{16}{3} - 4\right]$$

$$= 12 - 10 = \boxed{2}$$

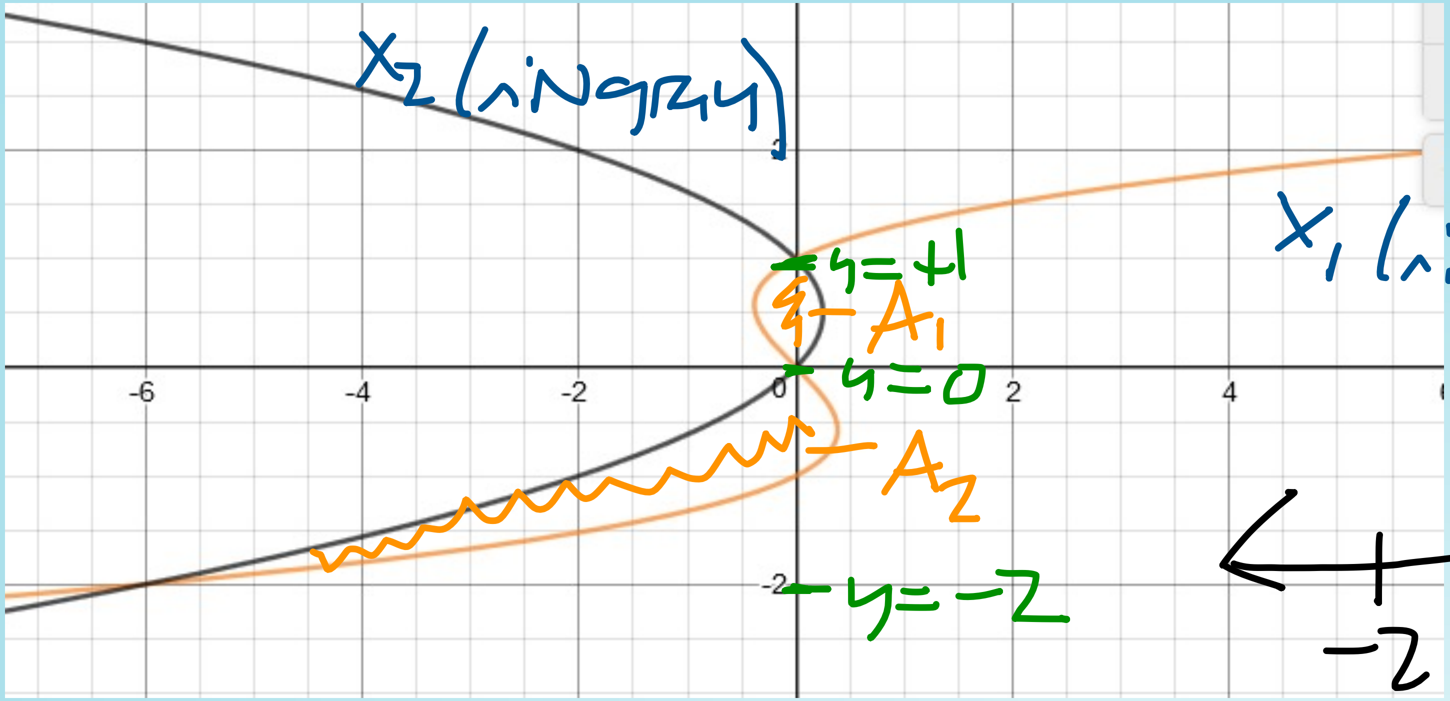
Example 2: → means for y in the interval [-2, 0]

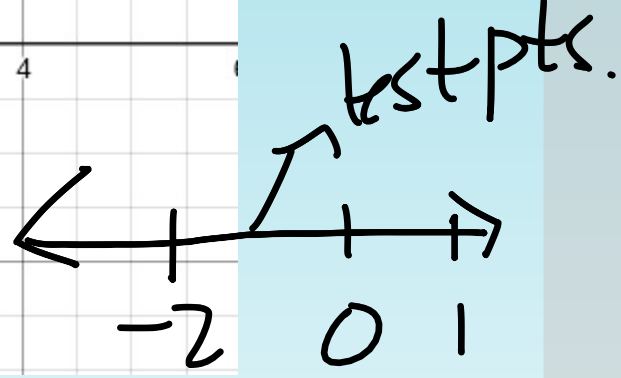
Find the area of the region bounded by

$$x+y-y^3=0 \text{ and } x-y+y^2=0.$$

Step 2:

- for $y \in [-2,0]$, $x_1 \geq x_2$
- for $y \in [0,1]$, $x_2 \geq x_1$





$x_1(y)=y^3-y$

$x_2(y)=y-y^2$

(*) Solved for int. points (below):
y = -2, 0, +1

Step 1: find the int. points (in y):
$(x_1 = x_2)$

$$y^3 - y = y - y^2 \Longleftrightarrow y^3 + y^2 - 2y = 0$$

$$\Longleftrightarrow y(y^2 + y - 2) \Longleftrightarrow y(y+2)(y-1) = 0$$

$\longrightarrow$ int points: $y = 0, +1, -2$ $(*)$

Step 3: setup integrals for A:

$x_1(y) = y^3 - y$
$x_2(y) = y - y^2$

area bdd. between x_1 and x_2:

"bounded"

$$A = \underbrace{\int_{-2}^{0} (x_1(y) - x_2(y))\,dy}_{A_1}$$

$$+ \underbrace{\int_{0}^{1} (x_2(y) - x_1(y))\,dy}_{A_2}$$

$$= \int_{-2}^{0} (y^3 - 2y + y^2)\,dy - \int_{0}^{1} (y^3 - 2y + y^2)\,dy$$

$$= \left(\frac{y^4}{4} - y^2 + \frac{y^3}{3}\right)\Bigg|_{-2}^{0} - \left(\frac{y^4}{4} - y^2 + \frac{y^3}{3}\right)\Bigg|_{0}^{1}$$

$$= \left[0 - \left(4 - 4 - \frac{8}{3}\right)\right] - \left[\left(\frac{1}{4} - 1 + \frac{1}{3}\right) - 0\right]$$

$$= +8/3 + 1 - \frac{1}{4} - \frac{1}{3}$$

$$= \frac{28}{12} + \frac{12}{12} - \frac{3}{12} = \boxed{\frac{37}{12}}$$

Example 3:

Find the area bounded by the curves

$x = y^2$ and $x = \sqrt{y}$. $\longrightarrow$ Solve for y_1, y_2 as functions of x

A. 0
B. 1/3
C. 2/3
D. 1

$y_1 = \sqrt{x}$

$y_2 = x^2$

Step 1: find int. points ($y_1 = y_2$):

$x^2 = \sqrt{x} \Longleftrightarrow x^4 - x = 0$

is the same as the last line

$\Longleftrightarrow x(x^3 - 1) = 0 \Longleftrightarrow x(x-1)(x^2+x+1)$ $\neq 0$

quadratic formula: To solve for the roots of $ax^2+bx+c=0$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$(a=1, b=1, c=1)$

$$x^2+x+1=0$$

$i=\sqrt{-1}$

$$\Leftrightarrow x=\frac{-1\pm\sqrt{-3}}{2}=\frac{-1\pm\sqrt{3}\cdot i}{2}$$

$\rightarrow$ complex roots (don't worry about these geometrically)

following from: $x(x-1)(x^2+x+1)=0$

$\rightarrow$ int points: $x=0,+1$

Recall:

$y_1=\sqrt{x}$

$y_2=x^2$

step2: find subints, and top vs. bottom curves on the subints.

- for $x\in[0,1]$, $y_1 > y_2$

Step 3: Set up an integral for A:

$$A = \int_0^1 (y_1(x) - y_2(x))\,dx$$

$$= \int_0^1 (\sqrt{x} - x^2)\,dx = \left(\frac{2}{3}x^{3/2} - \frac{x^3}{3}\right)\Bigg|_0^1$$

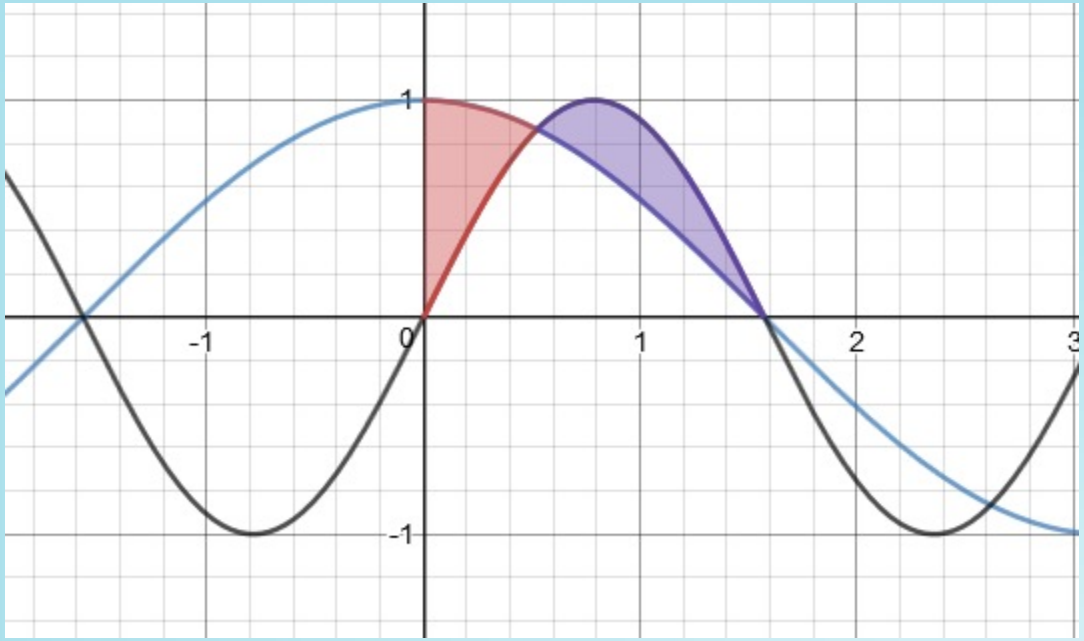
$$= \left(\frac{2}{3} - \frac{1}{3}\right) - 0 = \boxed{\frac{1}{3}}$$

Example 4:

Find the area of the region bounded by the curves

$y_1 = \cos x$ and $y_2 = \sin(2x)$ on $\left[0, \dfrac{\pi}{2}\right]$.

→ Start this today, finish on Friday



y_2 (in gray)

y_1 (in blue)

Subints: $[0, \pi/6]$ and $[\pi/6, \pi/2]$

step 1: find int. points of $y_1 = \cos x$ and $y_2 = \sin(2x)$ on $\left[0, \frac{\pi}{2}\right]$.

$y_1 = y_2$ for $x \in [0, \pi/2]$:

first recall that $\sin(2x) = 2\cos(x) \cdot \sin(x)$

$y_1 = y_2 \longleftrightarrow \cos(x) = 2\cos(x) \cdot \sin(x)$

$\rightarrow \frac{1}{2} = \sin(x) \longleftrightarrow \sin^{-1}(1/2) = x$

$= \pi/6$